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ABSTRACT

Rènyi (1959) defined the maximal correlation ρ between a pair of random variables (U, W) as

$$\sup\{\frac{cov(f(U), g(W))}{\sqrt{V(f(U))V(g(W))}} : V(f(U)) > 0, V(g(W)) > 0\},$$

where the supremum is taken over all functions of U and W with finite second moments. In this paper we derive the asymptotic distribution of the estimate of the Rènyi correlation coefficient based on a sample of independent observations under the assumption that (U, W) are independent and assume only a finite number of values.

1. INTRODUCTION

Rènyi (1959) defined the maximal correlation ρ between a pair of random variables (U, W) as

$$\sup\{\frac{cov(f(U), g(W))}{\sqrt{V(f(U))V(g(W))}} : V(f(U)) > 0, V(g(W)) > 0\},$$

where the supremum is taken over all functions of U and W with finite second moments. One of the attractive features of the Rènyi maximal correlation is that U and W are independent if and only if $\rho = 0$.

An explicit evaluation of the Rènyi maximal correlation is not available for a general random variable (U, W) except in very special cases. A case

of special interest is that of the bivariate normal distribution. The Rènyi maximal correlation for a bivariate normal distribution with correlation r is $|r|$, testifying to the fact that $r = 0$ implies independence. We will now give a direct evaluation of the Rènyi maximal correlation ρ when U and W take only finite number of values.

Suppose that U takes on only a finite number of values $\alpha_1, \ldots, \alpha_{r+1}$ and W takes on only a finite number of values $\beta_1, \ldots, \beta_{s+1}$. To avoid trivialities, we will assume that

$$\begin{aligned} P(U = \alpha_i) > 0, &\text{ for } 1 \leq i \leq r+1, \\ P(W = \beta_j) > 0, &\text{ for } 1 \leq j \leq s+1, \text{ and} \\ r \geq s. & \end{aligned} \tag{1}$$

In this case we can replace the bivariate random variable (U, W) by $\mathbf{Z} \stackrel{def}{=} \{\mathbf{X}, \mathbf{Y}\} \stackrel{def}{=} (X_1, \ldots, X_r, Y_1, \ldots, Y_s)'$ where $X_i = I(U = \alpha_i), 1 \leq i \leq r$ and $Y_j = I(W = \beta_j), 1 \leq j \leq s$, and where $I(\cdot)$ stands for the indicator function.

In the rest of this paper, we use the expression U and W *take on only a finite number of values* to mean what we have said above, including assumption (1).

It is clear that $(X_1, \ldots, X_r,\ Y_1, \ldots, Y_s)'$ is a one-to-one function of (U, W) and thus for all statistical purposes, the random variable (U, W) can be replaced by $\{\mathbf{X}, \mathbf{Y}\}$. Notice that in view of (1), $X_1, \ldots, X_r$ are linearly independent and X_{r+1} is a simple linear function of $X_1, \ldots, X_r$. Similarly $Y_1, \ldots, Y_s$ are linearly independent and Y_{s+1} is a simple linear function of $Y_1, \ldots, Y_s$. We can give an easy explicit form for the Rènyi maximal correlation of (U, W) in terms of the variance covariance matrix of $\{\mathbf{X}, \mathbf{Y}\}$. To do this we need to set up the following definitions.

Let $E(\mathbf{X}) = \gamma, E(\mathbf{Y}) = \delta, E((\mathbf{X}-\gamma)(\mathbf{X}-\gamma)') = \Gamma, E((\mathbf{Y}-\delta)(\mathbf{Y}-\delta)') = \Delta$, and $E((\mathbf{X}-\gamma)(\mathbf{Y}-\delta)') = \Theta$. Then Γ and Δ are the variance covariance matrices of $\mathbf{X}$ and $\mathbf{Y}$, respectively and Θ is the covariance matrix between $\mathbf{X}$ and $\mathbf{Y}$. We can rephrase (1) by saying that Γ and Δ are of full rank.

Notice that the most general function $f(U)$ of U is no more than a linear function $\mathbf{a}'\mathbf{X}$ of $\mathbf{X}$ for some vector $\mathbf{a}$. Similarly, the most general function $g(W)$ of W can be replaced by a linear function $\mathbf{b}'\mathbf{Y}$ of $\mathbf{Y}$ for some vector $\mathbf{b}$. Thus the maximal Rènyi correlation ρ is given by

$$\rho = \sup\{\mathbf{a}'\Theta\mathbf{b}/\sqrt{(\mathbf{a}'\Gamma\mathbf{a})\ (\mathbf{b}'\Delta\mathbf{b})} : \mathbf{a}'\Gamma\mathbf{a} > 0, \mathbf{b}'\Delta\mathbf{b} > 0\}. \tag{2}$$

Let M and N be nonsingular matrices such that $M'M = \Gamma$ and $N'N = \Delta$. We can simplify (2) to read as

$$\rho = \sup\{\mathbf{a}'(M')^{-1}\Theta N^{-1}\mathbf{b} : \mathbf{a}'\mathbf{a} = 1, \mathbf{b}'\mathbf{b} = 1\}. \tag{3}$$

From standard matrix manipulations, the maximization problem in (3) can be solved and we find that

$$\rho = \sqrt{\mu_1}$$

where $\mu_1 =$ maximum eigenvalue of $(N')^{-1}\Theta' M^{-1}(M')^{-1}\Theta N^{-1}$.

We may note that $\sqrt{\mu_1}$ is the first canonical correlation between $\mathbf{X}$ and $\mathbf{Y}$, as defined in the literature (*e. g.* Anderson (1958), p. 295). Canonical correlations can be defined for any two random vectors $\mathbf{X}$ and $\mathbf{Y}$ which need not consist just of indicator random variables as considered here.

2. ESTIMATION OF THE RÈNYI MAXIMAL CORRELATION BASED ON A SAMPLE

Suppose that we have a sample $\{(U_t, W_t), 1 \leq t \leq n\}$ of independent and identically distributed observations on (U, W). How should we estimate ρ and what will be the asymptotic distribution of this estimate? We propose to address these questions in this paper.

This problem does not seem to have an easy solution when (U, W) is a general bivariate random variable. However when U and W take on only a finite number of values as described in Section 1 and $\rho = 0$, we are able to give a solution to the questions posed above. The final result is given in Theorem 4 of Section 3. We announced this result in Sethuraman (1977).

We can replace $(U_1, W_1), \ldots, (U_n, W_n)$ by $(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = (\mathbf{X}_t, \mathbf{Y}_t), 1 \leq t \leq n$, where $X_{ti} = I(U_t = \alpha_i), Y_{tj} = I(W_t = \beta_j)$, $1 \leq t \leq n$, $1 \leq i \leq r$, $1 \leq j \leq s$, by using the method described in Section 1.

Let

$$\bar{X}_i = (1/n) \sum_{1 \leq t \leq n} X_{ti},$$

$$\bar{Y}_i = (1/n) \sum_{1 \leq t \leq n} Y_{ti},$$

$$c_{ii'} = (1/n) \sum_{1 \le t \le n} X_{ti} X_{ti'} - \bar{X}_i \bar{X}_{i'},$$

$$d_{ii'} = (1/n) \sum_{1 \le t \le n} Y_{ti} Y_{ti'} - \bar{Y}_i \bar{Y}_{i'}, \text{ and}$$

$$e_{ii} = (1/n) \sum_{1 \le t \le n} X_{ti} Y_{ti'} - \bar{X}_i \bar{Y}_{i'}.$$

The matrix $\begin{pmatrix} C & E \\ E' & D \end{pmatrix}$ represents the sample variance covariance matrix of $(\mathbf{X}_1, \mathbf{Y}_1), \ldots, (\mathbf{X}_n, \mathbf{Y}_n)$.

More generally, for $\mathbf{a} = (a_1, \ldots, a_r)' \in R_r$ and $\mathbf{b} = (b_1, \ldots, b_s)' \in R_s$, define

$$\bar{X}(\mathbf{a}) = (1/n) \sum_t (\sum_i a_i X_{ti}),$$

$$\bar{Y}(\mathbf{b}) = (1/n) \sum_t (\sum_j b_j Y_{tj}),$$

$$c(\mathbf{a}, \mathbf{a}) = (1/n) \sum_t (\sum_i a_i X_{ti})^2 - \bar{X}(\mathbf{a})^2,$$

$$d(\mathbf{b}, \mathbf{b}) = (1/n) \sum_t (\sum_i b_j Y_{tj})^2 - \bar{Y}(\mathbf{b})^2,$$

$$e(\mathbf{a}, \mathbf{b}) = (1/n) \sum_t (\sum_i a_i X_{ti})(\sum_j b_j Y_{tj}) - \bar{X}(\mathbf{a}) \bar{Y}(\mathbf{b}), \text{ and}$$

$$r(\mathbf{a}, \mathbf{b}) = \begin{cases} e(\mathbf{a}, \mathbf{b}) / \sqrt{c(\mathbf{a}, \mathbf{a})\ d(\mathbf{b}, \mathbf{b})} & \text{if the denominator} \ne 0 \\ 0 & \text{if the denominator} = 0\ . \end{cases}$$

That is $r(\mathbf{a}, \mathbf{b})$ is the sample estimate of $corr(\sum_i a_i X_i, \sum_j b_j Y_j)$. Let

$$r^* = \sup_{\mathbf{a}, \mathbf{b}} r(\mathbf{a}, \mathbf{b}).$$

Then r^* is the sample maximal linear correlation between $\mathbf{X}$ and $\mathbf{Y}$. It is also the sample Rènyi maximal correlation based on $(U_1, W_1) \ldots, (U_n, W_n)$. It is natural to use r^* as an estimate of ρ.

3. THE ASYMPTOTIC DISTRIBUTION OF THE SAMPLE RÈNYI CORRELATION COEFFICIENT

We continue to make the assumption that U and W take on only a finite number of values. Throughout this section we will make the additional

assumption that $\rho = 0$. Under these assumptions, we will obtain the asymptotic distribution of r^* in Theorem 4. Before proving this theorem we will establish some preliminary results.

Theorem 1. Let

$$S \stackrel{def}{=} \begin{pmatrix} C & \sqrt{n}E \\ \sqrt{n}E' & D \end{pmatrix} \stackrel{def}{=} (s_{k,k'}), \quad 1 \le k, k' \le r + s.$$

Then $S \to \Sigma$ in distribution, where $\Sigma = \begin{pmatrix} \Gamma & \Xi \\ \Xi' & \Delta \end{pmatrix}$, and the elements $\{\xi_{i,j}\}$ of Ξ, $1 \le i \le r$, $1 \le j \le s$, have a multivariate normal distribution with mean 0 and $cov(\xi_{ij}, \xi_{i'j'}) = \Gamma_{ii'}\Delta_{jj'}$, $1 \le i, i' \le r$, $1 \le j, j' \le s$.

Proof. Notice that all moments of $\mathbf{X}$ and $\mathbf{Y}$ are finite. Furthermore, $cov(X_i, X_{i'}) = \Gamma_{ii'}$ and $cov(Y_j, Y_{j'}) = \Delta_{jj'}$. This implies that

$$C_{ii'} \to \Gamma_{i,i'}, \quad D_{jj'} \to \Delta_{jj'}$$

in distribution (and also w.p. 1), $1 \le i, i' \le r$, $1 \le j, j' \le s$. Again, since $\rho = 0$, it follows that $cov(X_iY_j, X_{i'}Y_{j'}) = \Gamma_{ii'}\Delta_{jj'}$. Furthermore, $\sqrt{n}e_{ij}$ is the normalized sample mean of X_iY_j. From the multivariate central limit theorem, it follows that $\sqrt{n}E = \{\sqrt{n}e_{ij},\ 1 \le i \le r,\ 1 \le j \le s\}$ converges in distribution to Ξ which has a multivariate normal distribution with means 0 and with $cov(\xi_{ij}, \xi_{i'j'}) = \Gamma_{ii'}\Delta_{jj'}$, $1 \le i, i' \le r$, $1 \le j, j' \le s$. This completes the proof of the theorem. $\Diamond$

Note that the joint distribution of Ξ above can be stated more concisely as follows, using the vec and Kronecker product notations for matrices. The distribution of vec Ξ is multivariate normal with mean $\mathbf{0}$ and covariance matrix $\Gamma \otimes \Delta$.

Theorem 2. The limiting distribution of $\sqrt{n}r^*$ is the distribution of

$$\sup\{\mathbf{a}'\Xi\mathbf{b}/\sqrt{(\mathbf{a}'\Gamma\mathbf{a})\ (\mathbf{b}'\Delta\mathbf{b})} : \mathbf{a}'\Gamma\mathbf{a} > 0, \mathbf{b}'\Delta\mathbf{b} > 0\},$$

where Ξ has the distribution specified in Theorem 1.

Proof. Let $\mathcal{S} = \{$ all matrices of the type $\begin{pmatrix} C & E \\ E' & D \end{pmatrix}$ where C and D are positive definite matrices $\}$. Let the function f on $\mathcal{S}$ be defined as follows:

$$\begin{aligned} f\begin{pmatrix} C & E \\ E' & D \end{pmatrix} &= \sqrt{n}r^* \\ &= \sup_{\mathbf{a},\mathbf{b}} \sqrt{n}r(\mathbf{a}, \mathbf{b}) \\ &= \sup_{\mathbf{a},\mathbf{b}:\mathbf{a}'C\mathbf{a}>0,\mathbf{b}'D\mathbf{b}>0} \frac{\sqrt{n}\mathbf{a}'E\mathbf{b}}{\sqrt{(\mathbf{a}'C\mathbf{a})\ (\mathbf{b}'D\mathbf{b})}}. \end{aligned}$$

It is easy to see that if S_k is any sequence of nonnegative definite matrices such that $S_k \to \Sigma = \begin{pmatrix} \Gamma & \Xi \\ \Xi' & \Delta \end{pmatrix}$ pointwise, then $f(S_k) \to f(\Sigma)$. Thus from the invariance principle for functions of a convergent sequence of random variables, it follows that the limiting distribution of $\sqrt{n}r^*$ is the distribution of

$$\sup\{\mathbf{a}'\Xi\mathbf{b}/\sqrt{(\mathbf{a}'\Gamma\mathbf{a})\ (\mathbf{b}'\Delta\mathbf{b})} : \mathbf{a}'\Gamma\mathbf{a} > 0, \mathbf{b}'\Delta\mathbf{b} > 0\}. \qquad \diamond$$

Theorem 3. Let $\Xi_{r\times s}$ be a random matrix such that

$$\text{vec}\Xi \sim MN(\mathbf{0}, \Gamma \otimes \Delta).$$

Let $M'M = \Gamma$, and $N'N = \Delta$, where M, N are square matrices of order r and s and of full ranks. Then

$$\text{vec}((M')^{-1}\Xi N^{-1}) \sim MN(\mathbf{0}, I_r \otimes I_s),$$

and the distribution of

$$(N')^{-1}\Xi' M^{-1}(M')^{-1}\Xi N^{-1}$$

is the Wishart distribution $W(I_s, r)$.

Proof. This is easily proved by direct computation from one of the standard definitions of the Wishart distribution. See Anderson (1958), p. 157. $\diamond$

Theorem 4. The limiting distribution of $\sqrt{n}r^*$ is the distribution of $\sqrt{\lambda_1}$ where λ_1 is the maximum eigenvalue of W, where W has a Wishart distribution $W(I_s, r)$.

Proof. From Theorem 2, the limiting distribution of $\sqrt{n}r^*$ is the distribution of

$$\sup\{\mathbf{a}'\Xi\mathbf{b}/\sqrt{(\mathbf{a}'\Gamma\mathbf{a})\ (\mathbf{b}'\Delta\mathbf{b})} : \mathbf{a}'\Gamma\mathbf{a} > 0, \mathbf{b}'\Delta\mathbf{b} > 0\}.$$

Let $M'M = \Gamma$, and $N'N = \Delta$, where M, N are square matrices of order r and s and of full ranks as in Theorem 3. Then

$$\begin{aligned}
&\sup\{\mathbf{a}'\Xi\mathbf{b}/\sqrt{(\mathbf{a}'\Gamma\mathbf{a})\ (\mathbf{b}'\Delta\mathbf{b})} : \mathbf{a}'\Gamma\mathbf{a} > 0, \mathbf{b}'\Delta\mathbf{b} > 0\} \\
&= \sup\{\mathbf{a}'(M')^{-1}\Xi N^{-1}\mathbf{b}/\sqrt{(\mathbf{a}'\mathbf{a})\ (\mathbf{b}'\mathbf{b})} : \mathbf{a}'\mathbf{a} > 0, \mathbf{b}'\mathbf{b} > 0\} \\
&= \sup\{\mathbf{a}'(M')^{-1}\Xi N^{-1}\mathbf{b} : \mathbf{a}'\mathbf{a} = 1, \mathbf{b}'\mathbf{b} = 1\} \\
&= \sqrt{\text{max eigenvalue of } (N')^{-1}\Xi' M^{-1}(M')^{-1}\Xi N^{-1}}.
\end{aligned}$$

Now, from Theorem 3, $(N')^{-1}\Xi' M^{-1}(M')^{-1}\Xi N^{-1}$ has a Wishart distribution $W(I_s, r)$. Thus the limiting distribution of $\sqrt{n}r^*$ is the distribution of

$\sqrt{\lambda_1}$ where λ_1 is the maximum eigenvalue of W where W has a Wishart distribution $W(I_s, r)$. $\diamond$

Remark. Theorem 4 also establishes the asymptotic distribution of the sample canonical correlation coefficient based on two random vectors $\mathbf{X}$ and $\mathbf{Y}$ for which $corr(\mathbf{a}'\mathbf{X}, \mathbf{b}'\mathbf{Y}) = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$. Notice that we did not have to assume that $\mathbf{X}$ and $\mathbf{Y}$ have multivariate normal distributions.

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